FALL 2024 MATH 147 : MIDTERM EXAM II SOLUTIONS

When applicable, show all work to receive full credit. When in doubt, it is better to show more work than less.

Please work each problem on a separate sheet of paper, using the reverse side if necessary. Be sure to put your name on each page of your solutions. Good luck on the exam!

1. Short answer (30 points)

(i) Define $\int\int_R f(x,y)\ dA$, for R a closed and bounded region in $\mathbb{R}^2$.
(ii) Under what conditions on $f(x,y)$ is it reasonable to expect that the integral in (i) exists?
(iii) Find a partial sum for $\int\int_R 2x\ dA$ that approximates the true value to within 10^{-1}, for $R = [0,1]\times[0,1]$.
(iv) Let $B \subseteq \mathbb{R}^3$ be the solid in the first octant bounded by $x+2y+3z=1$, the xy-plane, the xz-plane, and the yz-plane. Set up the triple integral $\int\int\int_B 2xyz\ dV$ is three different ways, so that the orders of integration are: $dzdydx$; $dxdzdy$; $dydxdz$.
(v) Find the transformation $G(u,v)$ from the uv-plane to the xy-plane that takes the unit square in the uv-plane to the parallelogram P obtained by translating the lower left corner of the parallelogram P' spanned by the vectors $\vec{i}+2\vec{j}, 2\vec{i}+\vec{j}$ to the point (3,5). What is the Jacobian of $G(u,v)$?

Solution. (i) There are several possible ways to answer this question. The easiest: Subdivide R into small regions $R_1,\dots,R_n$, each of area ΔA. Pick a point $P_i \in R_i$. Then $\int\int_R f(x,y)\ dA = \lim_{n\to\infty}\Sigma_{i=1}^n f(P_i)\Delta A$.

Solution. For (i): Subdivide R into small pieces of area A_i, each of size ΔA. Choose $P_i \in A_i$. Form the sum $\sum_i f(P_i)\Delta A$. Take the limit at $\Delta A \to 0$. For (ii): The integral in (i) exists if $f(x,y)$ is continuous on R.

For (ii): If $f(x,y)$ is continuous on R, the integral in (i) exists.

For (iii): There are many answers. First note that $\int\int_R 2x\ dA = 1$. Subdivide each [0,1] into to intervals of equal sizes. This gives a subdivision of R into four squares of area $\frac{1}{4}$. If we evaluate $f(x,y)$ at each center point and multiply by $\frac{1}{4}$, we get the following partial sum:

$$f(\frac{1}{4},\frac{1}{4})\cdot\frac{1}{4}+f(\frac{1}{4},\frac{3}{4})\cdot\frac{1}{4}+f(\frac{3}{4},\frac{1}{4})\cdot\frac{1}{4}+f(\frac{3}{4},\frac{3}{4})\cdot\frac{1}{4} = (2\cdot\frac{1}{4})\cdot\frac{1}{4}+(2\cdot\frac{1}{4})\cdot\frac{1}{4}+(2\cdot\frac{3}{4})\cdot\frac{1}{4}+(2\cdot\frac{3}{4})\cdot\frac{1}{4} = 1.$$

For (iv): $\int_0^{\frac{1}{2}}\int_0^{\frac{1}{3}-\frac{2}{3}y}\int_0^{1-3z-2y} 2xyz\ dxdzdy = \int_0^{\frac{1}{3}}\int_0^{1-3z}\int_0^{\frac{1}{2}-\frac{3}{2}z-\frac{1}{2}x} 2xyz\ dydxdz = \int_0^1\int_0^{\frac{1}{2}-\frac{x}{2}}\int_0^{\frac{1}{3}-\frac{2}{3}y-\frac{1}{3}x} 2xyz\ dzdydz.$

For (v): $G(u,v) = (u+2v+3, 2u+v+5)$ and $\text{Jac}(G) = -3$.

2. Let B denote that portion of the solid ball of radius R centered at the origin that lies in the first octant. Calculate $\int\int\int_B xyz\ dV$. (20 points)

Solution. Using spherical coordinates we have:

$$\begin{aligned}
\int\int\int_B xyz\ dV &= \int_0^{\frac{\pi}{2}}\int_0^{\frac{\pi}{2}}\int_0^R (\rho\sin(\phi)\cos(\theta))(\rho\sin(\phi)\sin(\theta))(\rho\cos(\phi))\ \rho^2\sin(\phi)\ d\rho d\phi d\theta \\
&= \int_0^{\frac{\pi}{2}}\int_0^{\frac{\pi}{2}}\int_0^R \rho^5\sin^3(\phi)\cos(\phi)\cos(\theta)\sin(\theta)\ d\rho d\phi d\theta \\
&= \frac{R^6}{6}\int_0^{\frac{\pi}{2}}\int_0^{\frac{\pi}{2}} \sin^3(\phi)\cos(\phi)\cos(\theta)\sin(\theta)\ d\phi d\theta \\
&= \frac{R^6}{6}\cdot\{\int_0^{\frac{\pi}{2}}\sin^3(\phi)\cos(\phi)\ d\phi\}\cdot\{\int_0^{\frac{\pi}{2}}\sin(\theta)\cos(\theta)\ d\theta\} \\
&= \frac{R^6}{6}\cdot\{\frac{\sin^4(\phi)}{4}\}_0^{\frac{\pi}{2}}\cdot\{\frac{\sin^2(\theta}{2}\}_0^{\frac{\pi}{2}} \\
&= \frac{R^6}{6}\cdot\frac{1}{4}\cdot\frac{1}{2} = \frac{R^6}{48}.
\end{aligned}$$

3. Let D denote the region in $\mathbb{R}^2$ between the ellipses $E_1 : \frac{(x-4)^2}{9}+\frac{(y+7)^2}{16} = 1$ and $E_2 : \frac{(x-4)^2}{36}+\frac{(y+7)^2}{64} = 1$. Calculate $\int\int_D \sqrt{16(x-4)^2+9(y+7)^2}\ dA$. (20 points)

Solution. We will transform the ellipses to circles centered at the origin in the uv-plane, and then use polar coordinates. First use the transformation $G(u,v) = (3u+4, 4v-7) = (x,y)$, which has $\mathrm{Jac}(G) = 12$. Notice that if we substitute these equations into the equations for the ellipses, E_1 becomes the circle $C_1 : u^2+v^2 = 1$ and E_2 becomes the circle $C_2 : u^2+v^2 = 4$. In addition, the integrand becomes $12(u^2+v^2)^{\frac{1}{2}}$. Thus, if we let D_0 denote the region in the uv-plane between the circles C_1 and C_2, we have

$$\begin{aligned}
\int\int_D \sqrt{16(x-4)^2+9(y+7)^2}\ dA &= \int\int_{D_0} 12(u^2+v^2)^{\frac{1}{2}}\ 12\ dA \\
&= 144\int_0^{2\pi}\int_1^2 r\cdot r\ dr\ d\theta \\
&= 144\int_0^{2\pi} \left.\frac{r^3}{3}\right|_{r=1}^{r=2} d\theta \\
&= 144\cdot\frac{7}{3}\cdot 2\pi \\
&= 672\pi.
\end{aligned}$$

4. Let D be the unbounded region in $\mathbb{R}^2$ consisting of all points (x,y) such that $x^2+y^2 \geq 4$. Sketch a picture of D and then evaluate the improper integral $\int\int_D \frac{17}{(x^2+y^2+1)^3}\ dxdy$. (20 points)

Solution. Using polar coordinates,

$$\begin{aligned}
\int\int_D \frac{17}{(x^2+y^2+1)^3}\ dxdy &= \int_0^{2\pi}\int_2^{\infty} \frac{17}{(r^2+1)^3}\ rdrd\theta \\
&= 2\pi\int_2^{\infty}\frac{17}{(r^2+1)^3}\ rdr \\
&= 2\pi\lim_{b\to\infty}\int_2^b \frac{17}{(r^2+1)^3}\ rdr \\
&= 2\pi\lim_{b\to\infty} \left. -\frac{17}{4}\cdot(r^2+1)^{-2}\right|_2^b \\
&= 2\pi\lim_{b\to\infty}\{-\frac{17}{4}\cdot(b^2+1)^{-2}+\frac{17}{4}\cdot(2^2+1)^{-2}\} \\
&= \frac{17\pi}{50}.
\end{aligned}$$

5. Calculate $\int\int_R(\frac{2y^4}{x^2}+\frac{3y^2}{x})e^{xy}\ dA$, where R is the region in the xy-plane bounded by $y^2 = 3x$, $y^2 = 2x$, $xy = 2$, $xy = 1$. Hint: It's easier to work with the inverse transform $F(x,y)$ of $G(u,v)$ taking the xy-plane to the uv-plane. You can use the fact that $\mathrm{Jac}(G) = \frac{1}{\mathrm{Jac}(F)}$. (20 points)

Solution. Set $u = \frac{y^2}{x}$ and $v = xy$. Note that this defines the inverse $F(x,y)$ of the change of variables. We have to calculate $G(u,v)$ to solve the problem.

To find the domain of integration in the uv-plane: Note $y^2 = 2x$ implies $u = \frac{y^2}{x} = 2$. This means F carries the parabola $y^2 = 2x$ to the line $u = 2$. Therefore, G carries the line $u = 2$ to the parabola $y^2 = 2x$. Similarly, F carries the parabola $y^2 = 3x$ to the line $u = 3$.

Also: F carries the hyperbolas $xy = 1$ and $xy = 2$ to the lines $v = 1$ and $v = 2$. Thus $G(R_0) = R$, where R_0 is the rectangle $2 \leq u \leq 3$ and $1 \leq v \leq 2$.

$$\mathrm{Jac}(F) = \begin{vmatrix} -\frac{y^2}{x^2} & \frac{2y}{x} \\ y & x \end{vmatrix} = -\frac{3y^2}{x} = -3v.$$

Thus, $|\mathrm{Jac}(G)| = |-\frac{1}{3v}| = \frac{1}{3v}$.

Thus,

$$\int\int_R (2\frac{y^4}{x^2} + 3\frac{y^2}{x})e^{xy}\ dA = \int_1^2 \int_2^3 (2u^2+3u)e^v\ \frac{1}{3u} dudv$$
$$= \int_2^3 \int_1^2 (\frac{2}{3}u+1)e^v\ dudv$$
$$= \int_1^2 e^v(\frac{1}{3}u^2+u)\Big|_{u=2}^{u=3}\ dv$$
$$= \frac{8}{3}\int_1^2 e^v\ dv$$
$$= \frac{8}{3}(e^2-e).$$

6. Let B denote the solid ellipsoid $0 \le \frac{(x-1)^2}{a^2} + \frac{(y-1)^2}{b^2} + \frac{(z-1)^2}{c^2} \le 1$ centered at $(1,1,1)$. For 20 points, use what you know about improper single and double integrals to evaluate the improper integral

$$\int\int\int_B \ln\sqrt{\frac{(x-1)^2}{a^2} + \frac{(y-1)^2}{b^2} + \frac{(z-1)^2}{c^2}}\ dV.$$

You may use the fact that $\int x^2\ln(x)\ dx = \frac{x^3}{3}\cdot(\ln(x)-\frac{1}{3})$.

Solution. Since there is only one point at which the integrand is undefined, this integral can be calculated by reducing to a single improper integral. First, change the domain of integration to C, the solid sphere of radius one, centered at the origin, and then use spherical coordinates. Or this can be done all at once by setting

$$x = a\rho\sin(\phi)\cos(\theta)+1$$
$$y = b\rho\sin(\phi)\sin(\theta)+1$$
$$z = c\rho\sin(\phi)\sin(\theta)+1.$$

The Jacobian of the transformation is $abc\rho^2\sin(\phi)$. Thus,

$$\int\int\int_B \ln\sqrt{\frac{(x-1)^2}{a^2} + \frac{(y-1)^2}{b^2} + \frac{(z-1)^2}{c^2}}\ dV = \int\int\int_C \ln(\rho)\cdot abc\rho^2\sin(\phi)\ d\rho d\phi d\theta.$$
$$= 2\pi abc\int_0^\pi \int_0^1 \ln(\rho)\rho^2\sin(\phi)\ d\rho d\phi$$
$$= 4\pi abc\int_0^1 \ln(\rho)\rho^2\ d\rho$$
$$= 4\pi abc\lim_{t\to 0}\int_t^1 \rho^2\ln(\rho)\ d\rho$$
$$= 4\pi abc\lim_{t\to 0}(\frac{\rho^3}{3}\ln(\rho)-\frac{\rho^3}{9})\Big|_t^1$$
$$= 4\pi abc\lim_{t\to 0}\{(0-\frac{1}{9})-(\frac{t^3}{3}\ln(t)-\frac{t^3}{9})\}$$
$$= 4\pi abc(-\frac{1}{9}-0)$$
$$= -\frac{4}{9}\pi abc.$$

7. Let B denote the solid sphere of radius R centered at the origin in $\mathbb{R}^3$, and let $P=(0,0,R)$ denote the north pole. Find the average value of the distance of points $(x,y,z)\in B$ to the point P. Note: $(a+b)^3-(a-b)^3 = 6a^2b+2b^3$. (20 points)

Solution. We need to find the average value of the function $f(x,y,z) = \sqrt{x^2+y^2+(z-R)^2}$ over the domain B. So we first calculate

$$\begin{aligned}\iiint_B \sqrt{x^2+y^2+(z-R)^2}\,dV &= \int_0^{2\pi}\int_0^{\pi}\int_0^{R} \sqrt{(\rho\sin(\phi)\cos(\theta))^2+(\rho\sin(\phi)\sin(\theta))^2+(\rho\cos(\phi)-R)^2}\,\rho^2\sin(\phi)\,d\rho\,d\phi\,d\theta\\ &= 2\pi\int_0^{\pi}\int_0^{R}\sqrt{\rho^2+R^2-2\rho R\cos(\phi)}\rho^2\sin(\phi)\,d\rho\,d\phi\\ &= 2\pi\int_0^{R}\int_0^{\pi}\sqrt{\rho^2+R^2-2\rho R\cos(\phi)}\rho^2\sin(\phi)\,d\phi\,d\rho.\end{aligned}$$

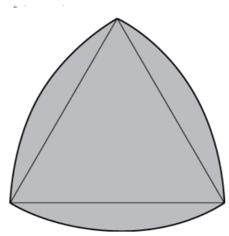
We can use u-substitution on the inner integral, by setting $u = \rho^2+R^2-2\rho R\cos(\phi)$. Then upon differentiating, $du = 2\rho R\sin(\phi)\,d\phi$, so that $\sin(\phi)\,d\phi = \frac{1}{2\rho R}du$. When $\phi = 0$, $u = (R-\rho)^2$ and when $\phi = \pi$, $u = (R+\rho)^2$, so continuing, we have

$$\begin{aligned}\iiint_B \sqrt{x^2+y^2+(z-R)^2}\,dV &= \int_0^{R}\int_{(R-\rho)^2}^{(R+\rho)^2}\sqrt{u}\rho^2\cdot\frac{1}{2\rho R}du\,d\rho\\ &= \frac{\pi}{R}\int_0^R\int_{(R-\rho)^2}^{(R+\rho)^2}\rho\sqrt{u}\,du\,d\rho\\ &= \frac{\pi}{R}\int_0^R \frac{2}{3}u^{\frac{3}{2}}\rho\Big|_{u=(R-\rho)^2}^{u=(R+\rho)^2}\,d\rho\\ &= \frac{2\pi}{3R}\int_0^R \rho\{(R+\rho)^3-(R-\rho)^3\}\,d\rho\\ &= \frac{2\pi}{3R}\int_0^R 6R^2\rho^2+2\rho^4\,d\rho\\ &= \frac{2\pi}{3R}\{2R^2\rho^3+\frac{2}{5}\rho^5\}_{\rho=0}^{\rho=R}\\ &= \frac{8}{5}\pi R^4.\end{aligned}$$

Thus,

$$\begin{aligned}\text{average distance to }(0,0,R) &= \frac{1}{\mathrm{vol}(B)}\iiint_B\sqrt{x^2+y^2+(z-R)^2}\,dV\\ &= \frac{3}{4\pi R^3}\cdot\frac{8}{5}\pi R^4\\ &= \frac{6R}{5}.\end{aligned}$$

Bonus Problem. The Reuleaux triangle consists of an equilateral triangle and three regions, each of them bounded by a side of the triangle and an arc of a circle of radius s centered at the opposite vertex of the triangle. Show that the area of the Reuleaux triangle in the following figure of side length s is $\frac{s^2}{2}(\pi-\sqrt{3})$. (20 points)



Solution. If we think of the lower left corner of the inner equilateral triangle as being at the origin, then the area of the first pie-shaped region is $\int_0^{\frac{\pi}{3}}\int_0^s r\,drd\theta = \frac{\pi s^2}{6}$. The area of three such regions is $\frac{\pi s^2}{2}$. These three regions cover the Reuleaux triangle, but in doing so, we have counted the area of the inner equilateral triangle three times. Since this area of the triangle is $\frac{\sqrt{3}s^2}{4}$, the area of the Reuleaux triangle is:

$$\frac{\pi s^2}{2} - 2 \cdot \frac{\sqrt{3}s^2}{4} = \frac{s^2}{2}(\pi - \sqrt{3}).$$